Exact Values and Bounds for Four Color Quadratic Polynomial van der Waerden Numbers

Ishan Kumthekar[1], William Gasarch[2], William Regli[2], and Ian Whitehouse[2]

[1]Department of Mathematics, University of Florida, Gainesville, FL 32611, USA,
ishankumthekar@ufl.edu
[2]Department of Computer Science, University of Maryland, College Park, MD 20742, USA,
gasarch@umd.edu, regli@umd.edu, ianjw@umd.edu

September 5, 2026

Abstract

For a nonzero polynomial $P(x) \in \mathbb{Z}[x]$ with $P(0) = 0$, let $W(P(x); 4)$ denote the least integer N such that every four coloring of $[N]$ contains distinct integers u, v of the same color with $v - u = P(d)$ for some integer d. The best known general upper bounds for polynomial van der Waerden numbers are primitive recursive but remain enormous compared with the few known exact values.

In this paper we determine the exact four color polynomial van der Waerden numbers for

$$\begin{array}{ccc} x^2 + x, & 2x^2 + x, & 3x^2 + x, \\ x^2 + 7x, & 2x^2 + 7x, & 3x^2 + 7x, \\ x^2 + 9x, & 2x^2 + 9x, & 3x^2 + 11x. \end{array}$$

We derive upper bounds using combinatorial proofs and lower bounds using constructions of explicit colorings of $[N]$ found using the alternative computing inspired method of simulated annealing.

More generally, we also prove that the four color polynomial van der Waerden numbers for the entire family of nonzero polynomials $ex^2 + fx$ grow only linearly in the magnitude of their coefficients

$$W(ex^2 + fx; 4) = \Theta(|e| + |f|),$$

1 Introduction

1.1 Van der Waerden numbers

Theorem 1.1 (van der Waerden [1]). *For any integers $r, \ell \geq 2$, there is an integer N such that every r coloring of $[N] = \{1, \ldots, N\}$ contains*

$$a,\ a + d,\ \ldots,\ a + (\ell - 1)d$$

of one color, for some $d > 0$.

Let $w(r, \ell)$ be the least such N. The upper bounds obtained from van der Waerden's original proof were not primitive recursive. Shelah [2] improved on them by obtaining primitive recursive bounds, and Gowers [3] later proved the explicit upper bound

$$w(r, \ell) \leq 2^{2^{r^{2^{2^{\ell+9}}}}}.$$

Exponential lower bounds are also known for fixed r as ℓ grows [4]. The seven known nontrivial exact values are [5, 6, 7]:

(r,ℓ)	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$	$(3,3)$	$(3,4)$	$(4,3)$
$w(r,\ell)$	9	35	178	1132	27	293	76

The known exact values are far smaller than the general upper bounds. Our quadratic results give another example of this gap and determine the dependence on the polynomial coefficients up to constant factors.

1.2 Polynomial van der Waerden numbers

The polynomial van der Waerden theorem replaces equally spaced terms by polynomial increments. Bergelson and Leibman proved the following theorem:

Theorem 1.2 (Bergelson and Leibman [12])**.** *Let $P_1, \ldots, P_m \in \mathbb{Z}[x]$ satisfy $P_i(0) = 0$. For any number of colors r, every r coloring of a sufficiently long interval $[N]$ contains*

$$a,\ a + P_1(d),\ \ldots,\ a + P_m(d)$$

of one color for some positive integer d.

This original proof established the finiteness of these numbers but did not provide explicit upper bounds on their values. Walters [8] subsequently gave a combinatorial proof with bounds that were not primitive recursive, while Shelah [9] obtained primitive recursive bounds that remain extremely large. The classical van der Waerden theorem is itself a special case of the polynomial theorem. Taking the polynomials $x, 2x, \ldots, (\ell-1)x$ gives an arithmetic progression of length ℓ. Thus the large gap between exponential lower bounds and known upper bounds is also present in polynomial van der Waerden numbers which we define as follows:

Definition 1.3. For a nonzero $P(x) \in \mathbb{Z}[x]$ with $P(0) = 0$ and $r \geq 2$, the *r color polynomial van der Waerden number* of P, denoted $W(P(x); r)$, is the least positive integer N such that every r coloring of $[N]$ contains distinct equally colored integers u, v with $v - u = P(d)$ for some $d \in \mathbb{Z}$.

Equivalently, let

$$\mathcal{D}(P) = \{|P(d)| : d \in \mathbb{Z},\ P(d) \neq 0\}.$$

For instance, $\mathcal{D}(x^2 + x) = \{2, 6, 12, 20, 30, 42, \ldots\}$. Join $u, v \in [N]$ when $|u - v| \in \mathcal{D}(P)$. Then $W(P(x); 4)$ is the first N for which this distance graph requires at least five colors. Thus each exact value determines a threshold for 4-colorability for an explicit family of finite graphs.

1.3 Prior results

Gasarch et al. [13] proved

$$W\left(x^2; 4\right) \leq 84{,}149{,}474{,}894{,}213{,}522, \qquad W\left(x^2 + x; 4\right) \leq 12{,}828{,}393{,}907,$$
$$W\left(x^2 + 2x; 4\right) \leq 66{,}287{,}711{,}296, \qquad W\left(x^2 + 3x; 4\right) \leq 10{,}952{,}789.$$

These bounds improve on the general proofs for the four quadratic polynomials shown above. Canakci et al. [14] later used SAT solvers to prove $W(x^2; 4) = 58$.

1.4 Our contribution

We investigate simulated annealing as a method for discovering exact polynomial van der Waerden numbers. For $x^2 + x$, $2x^2 + x$, and $3x^2 + x$, it found colorings of lengths 96, 48, and 32. Repeated attempts to extend these colorings were unsuccessful, suggesting the exact values 97, 49, and 33. We then proved these conjectures using a single lemma involving 17 integers. This demonstrates how probabilistic search methods can guide exact proofs. We also establish six further exact values by examining the restrictions imposed by polynomial differences. For example, for $2x^2+7x$, triangles in the distance graph force integers 12 apart to have the same color. These constraints lead to a contradiction on [25].

Finally, we establish bounds that apply to every nonzero polynomial $ex^2 + fx$, regardless of the size of its coefficients:

$$\frac{4}{9}(|e| + |f|) + 1 \leq W\Big(ex^2 + fx; 4\Big) \leq 64(|e| + |f|) + 1.$$

The main idea behind our upper bound is that we do not need a separate computation for every polynomial. For almost all choices of e and f, a four coloring of a sufficiently long interval would give a four coloring of the same graph on 715 vertices. We prove computationally that this graph cannot be four colored and handle the exceptional cases separately. We also give methods for constructing four colorings that establish lower bounds of the same asymptotic order. Consequently,

$$W\Big(ex^2 + fx; 4\Big) = \Theta(|e| + |f|).$$

This shows that the true growth for this entire family is linear in the coefficient sizes, despite the enormous general upper bounds from the polynomial van der Waerden theorem.

2 Exact Values

We use the following identity from Lemma 4 of Gasarch et al. [13].

Proposition 2.1. *For every positive integer h and every $r \geq 2$,*

$$W(hP(x); r) = h\big(W(P(x); r) - 1\big) + 1.$$

The same elementary fact is used in the first three upper bounds.

Lemma 2.2. *Every four coloring of*

$$\{0, 1, \ldots, 16\}$$

contains two equally colored integers whose difference belongs to

$$\mathcal{F} = \{1, 2, 5, 7, 12, 15\}.$$

Proof. Suppose, for contradiction, that there is a four coloring of

$$\{0, 1, \ldots, 16\}$$

with no two equally colored integers at a difference in $\mathcal{F}$.

There are 17 integers and only four colors. If every color were used at most four times, then at most $4 \cdot 4 = 16$ integers would be colored. Hence some color is used at least five times. Choose five integers of that color and write them in increasing order as

$$a_0 < a_1 < a_2 < a_3 < a_4.$$

For $1 \leq i \leq 4$, let $g_i = a_i - a_{i-1}$.

By our assumption, no two equally colored integers may differ by an element of $\mathcal{F}$. Since a_{i-1} and a_i have the same color, we must have

$$g_i \notin \{1, 2, 5, 7\}.$$

In particular, $g_i \geq 3$.

Also,

$$g_1 + g_2 + g_3 + g_4 = a_4 - a_0 \leq 16.$$

If some $g_i \geq 8$, then the other three gaps are each at least 3, so

$$g_1 + g_2 + g_3 + g_4 \geq 8 + 3 + 3 + 3 = 17,$$

a contradiction. Therefore every $g_i \leq 7$.

Hence each gap is an integer between 3 and 7, but it cannot equal 5 or 7. Therefore

$$g_i \in \{3, 4, 6\} \qquad (1 \leq i \leq 4).$$

We now consider how many of the four gaps are equal to 6. Suppose first that no gap is 6. Then every gap is either 3 or 4. A 3 and a 4 cannot occur next to one another, since two chosen positions would then differ by $3 + 4 = 7$, a forbidden distance. Hence all four gaps are equal. If they are all 3, then $a_4 - a_0 = 3 + 3 + 3 + 3 = 12$, while if they are all 4, then $a_3 - a_0 = 4 + 4 + 4 = 12$. Both cases give two equally colored positions at the forbidden difference 12.

Next, there cannot be two gaps equal to 6, since the remaining two gaps are at least 3, giving $g_1 + g_2 + g_3 + g_4 \geq 6 + 6 + 3 + 3 = 18 > 16$.

It remains to consider exactly one gap equal to 6. If the other three gaps are all 3, then $a_4 - a_0 = 6 + 3 + 3 + 3 = 15$, a forbidden distance. Otherwise one of the remaining gaps is 4. There cannot be two gaps equal to 4, since then $6 + 4 + 4 + 3 = 17 > 16$. Thus the four gaps are $6, 4, 3, 3$ in some order. Since a gap 4 cannot be adjacent to a gap 3, the 4 must be next to the 6 and at one end of the sequence. Up to reversing the order, the gaps are therefore

$$4, 6, 3, 3.$$

The final three gaps sum to $6 + 3 + 3 = 12$, so again two equally colored chosen positions differ by the forbidden distance 12.

All possibilities lead to a contradiction. This proves the lemma. □

Lemma 2.2 gives a sufficient condition for a short exact proof. If h is a positive integer with $h\mathcal{F} \subseteq \mathcal{D}(P)$, the colors of $1, 1 + h, \ldots, 1 + 16h$ would contradict the lemma. Thus $W(P(x); 4) \leq 16h + 1$. A valid four coloring of $[16h]$ then proves equality. The next three theorems use $h = 6, 3, 2$, respectively.

Theorem 2.3.

$$\boxed{W\Big(x^2 + x; 4\Big) = 97.}$$

Proof. **Upper bound.** Suppose that [97] has a valid four coloring for $P(x) = x^2 + x$. Consider the 17 integers

$$1, 7, 13, \ldots, 97.$$

These are exactly the numbers $1 + 6j$ for $0 \leq j \leq 16$. Give the index j the color of $1 + 6j$. By Lemma 2.2, there are indices $i < j$ with the same color and

$$j - i \in \mathcal{F}.$$

Therefore the corresponding integers in [97] have the same color and differ by

$$6(j - i) \in 6\mathcal{F} = \{6, 12, 30, 42, 72, 90\}.$$

Each of these is a value of $d^2 + d$:

$$6 = 2^2 + 2, \quad 12 = 3^2 + 3, \quad 30 = 5^2 + 5,$$
$$42 = 6^2 + 6, \quad 72 = 8^2 + 8, \quad 90 = 9^2 + 9.$$

Thus the assumed coloring contains an equally colored pair at a forbidden polynomial distance, a contradiction. Hence

$$W\left(x^2 + x; 4\right) \leq 97.$$

Lower bound. The nonzero distances in $\mathcal{D}(x^2 + x)$ that are smaller than 97 are

$$2, 6, 12, 20, 30, 42, 56, 72, 90.$$

The following coloring of [96] was found using simulated annealing. For each of the nine forbidden distances listed above, comparing every pair of entries at that separation shows that the two entries have different colors.

RRGGRRYYRRYYGGBBGGRRGGRRBBRRBBGGYYGGRRBBRRBBRRBB
GGYYGGYYGGYYBBRRBBGGYYGGYYBBYYBBGGBBRRYYRRYYBBYY

Hence a valid four coloring exists on [96], so

$$W\left(x^2 + x; 4\right) \geq 97. \qquad \square$$

Theorem 2.4.

$$\boxed{W\left(2x^2 + x; 4\right) = 49.}$$

Proof. **Upper bound.** Apply Lemma 2.2 to the colors of $1 + 3j$ for $0 \leq j \leq 16$, as in the preceding proof. An equally colored pair has difference in

$$3\mathcal{F} = \{3, 6, 15, 21, 36, 45\}.$$

For $P(x) = 2x^2 + x$, these are the values at $d = 1, -2, -3, 3, 4, -5$, respectively. They are all forbidden, so $W\left(2x^2 + x; 4\right) \leq 49$.

Lower bound. The values

$$P(m) = \frac{(2m)(2m+1)}{2}, \qquad P(-m) = \frac{(2m-1)(2m)}{2} \qquad (m \geq 1)$$

give every positive triangular number. The forbidden distances below 49 are $1, 3, 6, 10, 15, 21, 28, 36, 45$. The following coloring of [48] was found using simulated annealing. Direct comparison at each listed distance shows that no forbidden pair has the same color.

RGRYRYGBGRGRBRBGYGRBRBRBGYGYGYBRBGYGYBYBGBRYRYBY

Thus $W\left(2x^2 + x; 4\right) \geq 49$. $\square$

Theorem 2.5.

$$\boxed{W\left(3x^2 + x; 4\right) = 33.}$$

Proof. **Upper bound.** This time apply Lemma 2.2 to the colors of $1 + 2j$ for $0 \leq j \leq 16$. The resulting difference belongs to

$$2\mathcal{F} = \{2, 4, 10, 14, 24, 30\}.$$

For $P(x) = 3x^2 + x$, these values occur at $d = -1, 1, -2, 2, -3, 3$. Hence $W\left(3x^2 + x; 4\right) \leq 33$.

Lower bound. The same six values are all the forbidden distances below 33, since $|3d^2 + d| \geq 3d^2 - |d| \geq 44$ for $|d| \geq 4$. The following coloring of $[32]$ was found using simulated annealing. Entries at each of the six listed distances have different colors.

$$\boxed{\texttt{GGBBRRGGYYRRGGYYRRBBYYRRBBYYGGBB}}$$

Therefore $W(3x^2 + x; 4) \geq 33$. □

2.1 Odd cycles

Join two integers when their distance is forbidden. The next lemma is used in both upper bound proofs in this subsection.

Lemma 2.6. *In a proper four coloring, two vertices adjacent to every vertex of the same odd cycle have the same color.*

Proof. The cycle needs at least three colors. Each additional vertex excludes its color from the cycle, so both must use the remaining fourth color. □

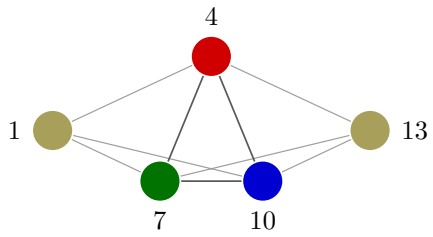

Figure 1: A common triangle for $2x^2 + 7x$. Every drawn edge has forbidden length 3, 6, or 9. The vertices $4, 7, 10$ use three colors, so 1 and 13 have the same fourth color.

Theorem 2.7.

$$\boxed{W\Big(2x^2 + 7x; 4\Big) = 25.}$$

Proof. **Upper bound.** The forbidden distances below 25 are $3, 4, 5, 6, 9, 15, 22$. Suppose c is a valid four coloring of $[25]$. For $1 \leq i \leq 13$, the vertices $i + 3, i + 6, i + 9$ form a triangle. Both i and $i + 12$ are adjacent to all three. Lemma 2.6 gives

$$c(i) = c(i + 12) \qquad (1 \leq i \leq 13).$$

Arrange the positions $1, \ldots, 12$ around a circle. Two positions separated by three, four, five, or six steps around this circle have different colors. Indeed, suitable representatives in $[24]$ differ by one of $3, 4, 5, 6$. Thus each color appears at most three times, and three positions of one color must be consecutive around the circle.

The color of position 1 cannot appear three times. The possible triples would be $(11, 12, 1)$, $(12, 1, 2)$, and $(1, 2, 3)$. They are ruled out, respectively, by the forbidden pairs $(1, 23)$, $(2, 24)$, and $(3, 25)$, each at distance 22, together with the equalities already proved. Hence the four colors cover at most $2 + 3 + 3 + 3 = 11$ of the twelve positions, a contradiction.

Lower bound. The following coloring of [24] was recovered by simulated annealing. Direct comparison at the seven listed distances gives no monochromatic forbidden pair.

$$\boxed{\text{YYYRRRGGGBBBYYYRRRGGGBBB}}$$

Thus $W(2x^2+7x;4) \geq 25$. □

Theorem 2.8.

$$\boxed{W\Big(2x^2+9x;4\Big)=29.}$$

Proof. **Upper bound.** The forbidden distances below 29 are $4,5,7,9,10,11,18,26$. Suppose c is a valid four coloring of $[29]$. For $5 \leq i \leq 11$, the vertices

$$i-4, \quad i+5, \quad i+9, \quad i+18, \quad i+7$$

form a five cycle in the displayed order. Its edge lengths are $9,4,9,11,11$. Both i and $i+14$ are adjacent to every vertex of this cycle, so Lemma 2.6 gives

$$c(i)=c(i+14) \qquad (5 \leq i \leq 11).$$

Now let $1 \leq i \leq 4$ and consider the colors at

$$i+4, \quad i+7, \quad i+10, \quad i+5, \quad i+9.$$

The first two consecutive pairs have different colors because $c(i+4)=c(i+18)$ and $c(i+7)=c(i+21)$, while the pairs $(i+7,i+18)$ and $(i+10,i+21)$ have forbidden distance 11. The other three consecutive pairs have forbidden distances $5,4,5$. These inequalities form a five cycle, so at least three colors occur. Both i and $i+14$ are adjacent to all five displayed vertices, so their colors agree. Reflection by $v \mapsto 30-v$ gives $12 \leq i \leq 15$. Thus

$$c(i)=c(i+14) \qquad (1 \leq i \leq 15).$$

Finally, consider the colors of $1,3,6,10,13$. The pairs $(1,6)$, $(1,10)$, $(3,10)$, $(3,13)$, $(6,10)$, and $(6,13)$ have different colors directly from their forbidden distances. The remaining four inequalities follow from the established equalities and these edges:

Colors compared	Forbidden pair	Distance
$c(1), c(3)$	$29, 3$	26
$c(1), c(13)$	$1, 27$	26
$c(3), c(6)$	$17, 6$	11
$c(10), c(13)$	$24, 13$	11

Five distinct colors are required, a contradiction.

Lower bound. The following annealed coloring of [28] has no monochromatic pair at any of the eight listed distances.

$$\boxed{\text{YYYRRRRBBBGGGGYYYYRRRBBBBGGG}}$$

Thus $W(2x^2+9x;4) \geq 29$. □

Proposition 2.9. *For every odd integer f and every $r \geq 2$,*

$$W\Big(x^2 + fx; r\Big) = 2W\Big(2x^2 + fx; r\Big) - 1.$$

Proof. Set $P(x) = x^2 + fx$ and $Q(x) = 2x^2 + fx$. For even $d = 2t$, $P(d) = 2Q(t)$. For odd d, the integer $t = -(d + f)/2$ also satisfies $P(d) = 2Q(t)$. Conversely, $2Q(t) = P(2t)$. Thus $\mathcal{D}(P) = 2\mathcal{D}(Q)$, and Proposition 2.1 gives the result. □

In particular, Theorems 2.7 and 2.8 give

$$\boxed{W\Big(x^2 + 7x; 4\Big) = 49, \qquad W\Big(x^2 + 9x; 4\Big) = 57.}$$

2.2 Block colorings

Theorem 2.10.

$$\boxed{W\Big(3x^2 + 7x; 4\Big) = 41, \qquad W\Big(3x^2 + 11x; 4\Big) = 17.}$$

Proof. **Upper bound.** Restrict a coloring of $[41]$ to its 21 odd integers, written as $1 + 2j$ for $0 \leq j \leq 20$. The distances $1, 2, 3, 20$ between the corresponding values of j are forbidden, since $2, 4, 6, 40 \in \mathcal{D}\big(3x^2 + 7x\big)$. Six values of j of one color would have five consecutive gaps of at least 4. Their total is at most 20, so all five gaps would be 4. The first and last would differ by the forbidden distance 20. Thus each color covers at most five of the 21 positions, a contradiction.

Lower bound. Repeat the four blocks **RR**, **GG**, **BB**, **YY** five times:

$$\boxed{\textbf{RRGGBBYYRRGGBBYYRRGGBBYYRRGGBBYYRRGGBBYY}}$$

The forbidden distances below 41 are $2, 4, 6, 10, 20, 26, 40$. An equally colored pair in this word has distance 1 or a distance in $[7, 9]$, $[15, 17]$, $[23, 25]$, or $[31, 33]$. None is forbidden, so the coloring of $[40]$ is valid.

Upper bound. Restrict $[17]$ to its nine odd integers. After dividing their differences by two, the forbidden distances are $2, 3, 4, 5, 7$. Three positions of one color would have two consecutive gaps from $\{1, 6, 8\}$ with sum at most 8. The only possibilities are $(1, 1)$, $(1, 6)$, and $(6, 1)$. Their sums 2 and 7 are forbidden. Each color therefore covers at most two of the nine positions, again a contradiction.

Lower bound. The smallest positive distance in $\mathcal{D}\big(3x^2 + 11x\big)$ is 4. Four constant blocks of length 4 give a valid coloring of $[16]$:

$$\boxed{\textbf{RRRRGGGGBBBBYYYY}}$$

Every equally colored pair has distance at most 3. Figure 2 shows all the forbidden pairs. □

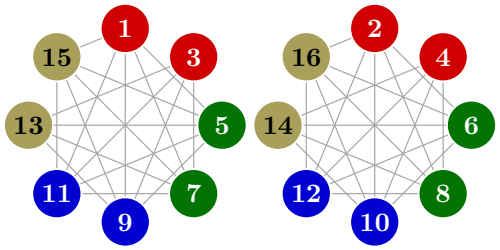

Figure 2: A valid coloring of $[16]$ for $3x^2+11x$. The 38 edges have lengths $4, 6, 8, 10, 14$, and every edge has differently colored endpoints. All forbidden distances are even, so the odd and even vertices form separate components.

3 Asymptotic Lower Bounds

The exact values above depend on the forbidden distances of individual polynomials. We now construct valid colorings that give lower bounds without solving each coloring problem separately.

Proposition 3.1. *Let $s_1 < s_2$ be the two smallest members of $\mathcal{D}(P)$. Then*

$$\boxed{W(P(x);4) \geq \max\{4s_1, 2s_2\} + 1.}$$

Proof. Color four consecutive blocks of length s_1 with four different colors. An equally colored pair lies in one block and has distance less than s_1. This gives a valid coloring of $[4s_1]$.

For the other bound, split $[2s_2]$ into two intervals of length s_2. Within either interval, the only possible forbidden distance is s_1. These edges form disjoint paths. Color each path alternately with two colors. Use a different pair of colors on the second interval. □

Proposition 3.2. *Suppose $h, J \geq 1$ are integers, no forbidden distance is smaller than h, and, for $1 \leq j < J$, no forbidden distance lies in*

$$[(4j-1)h+1, (4j+1)h-1].$$

Then

$$W(P(x);4) \geq 4hJ+1.$$

Proof. Repeat J times the word consisting of h copies of each of **R**, **G**, **B**, **Y**, in that order. An equally colored pair in the same block has distance at most $h-1$. If its blocks are $4j$ blocks apart, its distance lies in the displayed interval. Both possibilities are excluded by the hypotheses. □

The choices $(h, J) = (3, 2), (2, 5), (4, 1)$ give the exact lower bounds for $2x^2+7x$, $3x^2+7x$, and $3x^2+11x$, respectively.

3.1 Lower bounds in terms of the coefficients

The first constructions depend on the smallest forbidden distances. For quadratic polynomials we can also give lower bounds directly in terms of the coefficients, using the following consequence of a theorem of Barajas and Serra [11, Theorem 5].

Lemma 3.3. *Let $s_1 < s_2 < s_3 < s_4$ be positive integers with $s_3 > s_1 + s_2$. The integers have a four coloring avoiding all four distances.*

Proof. After division by their greatest common divisor, every set of four distances requiring five colors has its third smallest distance equal to the sum of the first two [11, Theorem 5]. The strict inequality excludes this case. A coloring for the normalized distances extends along each residue class of the common divisor. □

Theorem 3.4. *For $E, F > 0$, the following lower bounds hold:*

$$\begin{aligned} W(Ex^2 + Fx; 4) &\geq 9E - 3F + 1, && 0 < F < E, \\ W(Ex^2 + Fx; 4) &\geq 4E + 2F + 1, && E < F < 2E, \\ W(Ex^2 + Fx; 4) &\geq \left\lceil \frac{4(F-E)}{3} \right\rceil + 1, && F \geq E. \end{aligned}$$

Proof. If $0 < F < E$, the first five distinct forbidden distances are

$$E - F,\ E + F,\ 4E - 2F,\ 4E + 2F,\ 9E - 3F.$$

The third exceeds the sum of the first two. Lemma 3.3 therefore gives a four coloring avoiding the first four. Restrict it to $[9E - 3F]$, where no other forbidden distance can occur.

If $E < F < 2E$, write $\delta = F - E$, so $0 < \delta < E$. Below $4E + 2F = 6E + 2\delta$, the only possible forbidden distances are

$$\delta, \quad 2E - 2\delta, \quad 2E + \delta, \quad 6E - 3\delta.$$

Indeed, the next positive input value is $6E+2\delta$, and the next negative input value is $12E-4\delta > 6E+2\delta$. The first two displayed quantities are the smallest. If they coincide, there are at most three distinct distances, and greedy coloring uses four colors. Otherwise, the third exceeds their sum. Apply Lemma 3.3 and restrict to $[4E + 2F]$.

For the final bound, write $F = E(n + \rho)$ with $n \geq 1$ an integer and $0 \leq \rho < 1$. All positive forbidden distances except possibly

$$En\rho \quad \text{and} \quad E(n+1)(1-\rho)$$

are at least $F - E$. To check the negative inputs $-m$ with $1 \leq m < n$, the function $m(F - Em)$ is concave. Its endpoint values are $F - E$ and $E(n-1)(1+\rho)$, both at least $F - E$. When $n = 1$ this range is empty. Inputs with $m \geq n + 2$ give larger values, while positive inputs give at least $E + F$.

The only possible positive forbidden distances smaller than $F - E$ therefore arise from $d = -n$ and $d = -(n + 1)$. Their absolute values sum to

$$En\rho + E(n+1)(1-\rho) = E(n+1-\rho) \geq F - E.$$

Let $s_1 < s_2$ be the two smallest positive forbidden distances. If $s_1 \geq (F-E)/3$, then $4s_1 \geq 4(F-E)/3$. Otherwise, $s_1 < (F-E)/3$ must be one of the two exceptional distances above. Since their sum is at least $F-E$ and all other forbidden distances are at least $F-E$, we have $s_2 \geq (F-E)-s_1 > 2(F-E)/3$.

Thus, in either case,

$$\max\{4s_1, 2s_2\} \geq \frac{4(F-E)}{3}.$$

Proposition 3.1 now gives

$$W\Big(Ex^2 + Fx; 4\Big) \geq \left\lceil \frac{4(F-E)}{3} \right\rceil + 1.$$

□

Corollary 3.5. *For all integers e, f, not both zero,*

$$W(ex^2 + fx; 4) \geq \frac{4}{9}(|e| + |f|) + 1.$$

Proof. By symmetry, assume $E = |e|$ and $F = |f|$.
If $0 < F < E$, Theorem 3.4 gives

$$W(Ex^2 + Fx; 4) \geq 9E - 3F + 1 > \frac{4}{9}(E + F) + 1.$$

If $E < F < 2E$, the same theorem gives

$$W(Ex^2 + Fx; 4) \geq 4E + 2F + 1 > \frac{4}{9}(E + F) + 1.$$

If $F \geq 2E$, then

$$\frac{4}{3}(F - E) \geq \frac{4}{9}(E + F),$$

so the third bound of Theorem 3.4 implies the result.
Finally, if $E = 0$, $F = 0$, or $E = F$, the smallest positive forbidden distance is, respectively, F, E, or $2E$. Proposition 3.1 gives the claimed bound in each case. □

4 Asymptotic Upper Bound

For $P(x) = Ex^2 + Fx$ with $E, F \geq 0$, label a lattice point (u, v) by $\phi(u, v) = 1 + Fu + Ev$. Then

$$\phi(u + d, v + d^2) - \phi(u, v) = Ed^2 + Fd = P(d),$$
$$\phi(u + d, v - d^2) - \phi(u, v) = Fd - Ed^2 = -P(-d).$$

Thus pairs separated by $(d, \pm d^2)$ give polynomial differences. The following finite graph is used for every choice of the coefficients.

Lemma 4.1. *Let $Q = \{0, 1, \ldots, 10\} \times \{0, 1, \ldots, 64\}$. For $1 \leq d \leq 8$, join (u, v) to $(u + d, v + d^2)$ and to $(u + d, v - d^2)$ whenever both points lie in Q. Every four coloring of Q contains two equally colored joined points.*

Proof. The graph has 715 vertices and 4864 edges. The exact search finds no proper four coloring. The search procedure is described in Section 5. □

An edge of displacement $(d, -d^2)$ has label difference $d(F - Ed)$. If $F = dE$, its two endpoints receive the same integer label, so this edge cannot give a nonzero polynomial difference. Since $1 \leq d \leq 8$, the only excluded cases are $F = E, 2E, \ldots, 8E$.

Lemma 4.2. *For every $1 \leq j \leq 8$, $W(x^2 + jx; 4) \leq 97$.*

Proof. The case $j = 1$ is Theorem 2.3. For $j = 2, \ldots, 8$, the exact search is applied to [97], joining two integers when their distance is in $\mathcal{D}(x^2 + jx)$. In each of the seven cases the search finds no proper four coloring. □

Theorem 4.3. *For all integers e, f, not both zero,*

$$\boxed{W\Big(ex^2 + fx; 4\Big) \leq 64(|e| + |f|) + 1.}$$

Proof. Set $E = |e|$ and $F = |f|$. Replacing $P(x)$ by $-P(x)$ or $P(-x)$ does not change its forbidden distances, so assume $P(x) = Ex^2 + Fx$. If $E = 0$, the identity $W(Fx; 4) = 4F + 1$ from Gasarch et al. [13] gives the result. Hence assume $E > 0$.

First suppose $F \notin \{E, 2E, \ldots, 8E\}$. A valid coloring of $[64E + 10F + 1]$ would color every $(u, v) \in Q$ by the color of $1 + Fu + Ev$. Lemma 4.1 gives equally colored points separated by (d, d^2) or $(d, -d^2)$ for some $1 \leq d \leq 8$. Their integer labels differ by $P(d)$ or $-P(-d)$, respectively. The first is positive and the second is nonzero because $F \neq dE$. Either gives a forbidden equally colored pair, a contradiction. Thus

$$W\Big(Ex^2 + Fx; 4\Big) \leq 64E + 10F + 1 \leq 64(E + F) + 1.$$

If $F = jE$ for $1 \leq j \leq 8$, Lemma 4.2 and Proposition 2.1 give

$$W\Big(E(x^2 + jx); 4\Big) \leq 96E + 1 \leq 64(E + F) + 1,$$

since $j \geq 1$. This completes the proof. □

The known value $W(x^2; 4) = 58$ [14] and Proposition 2.1 give $W(Ex^2; 4) = 57E + 1$. Consequently, if C_4 is the smallest constant such that $W(ex^2 + fx; 4) \leq C_4(|e| + |f|) + 1$ for all nonzero pairs, then

$$57 \leq C_4 \leq 64.$$

Together with the coefficient lower bounds in Section 3, this establishes $W(ex^2 + fx; 4) = \Theta(|e| + |f|)$ for all nonzero integer coefficient pairs (e, f).

4.1 Comparison of Bounds to Exact Values

Our upper bound $64(|e| + |f|) + 1$ applies to every nonzero polynomial $ex^2 + fx$. Table 1 compares this bound with our nine exact values.

Polynomial	**Exact**	**UB**
$x^2 + x$	97	129
$2x^2 + x$	49	193
$3x^2 + x$	33	257
$2x^2 + 7x$	25	577
$x^2 + 7x$	49	513
$2x^2 + 9x$	29	705
$x^2 + 9x$	57	641
$3x^2 + 7x$	41	641
$3x^2 + 11x$	17	897

Table 1: Exact values and uniform upper bounds for our nine quadratic polynomials

For these nine polynomials, the upper bounds are within a factor of 53. Despite applying to every nonzero polynomial $ex^2 + fx$, these bounds provide reasonably close estimates in several cases. For example, our upper bound for $x^2 + x$ is 129, only 32 greater than the exact value 97.

The improvement over earlier explicit upper bounds is much greater. Gasarch et al. obtained upper bounds of 12,828,393,907 for $x^2 + x$ and approximately 6.57×10^{18} for $3x^2 + x$. Our bound reduces these to 129 and 257, respectively. Thus, our bounds not only establish the correct linear dependence on the coefficients but also give numerically useful estimates that are substantially closer to the exact values than earlier general estimates.

5 Computational Methods

5.1 Simulated annealing

For a polynomial $P(x)$ and interval $[N]$, let $E_P(N)$ be the set of forbidden pairs. For a four coloring $c : [N] \to \{1, 2, 3, 4\}$, define

$$E(c) = \#\{\{u, v\} \in E_P(N) : c(u) = c(v)\}.$$

Thus $E(c)$ is the number of monochromatic forbidden pairs, and $E(c) = 0$ exactly when c is a valid four coloring.

The energy change from recoloring one vertex depends only on its neighbors. We use the Metropolis simulated annealing rule [15]: a recoloring that does not increase E is accepted, while one increasing it by $\Delta > 0$ is accepted with probability $e^{-\Delta/T}$. We decrease T geometrically during each run.

Algorithm 1 Simulated annealing for the lower bounds

Input: Forbidden pairs $E_P(N)$, restarts R, proposals L, temperatures $T_0, T_{\min}$

Set $\alpha \leftarrow (T_{\min}/T_0)^{1/L}$
for $j = 1, \ldots, R$ **do**
  Choose a random four coloring c and set $T \leftarrow T_0$
  if $E(c) = 0$ **then**
    return c
  end if
  for $s = 1, \ldots, L$ **do**
    Choose a vertex v and a color $q \neq c(v)$ uniformly at random
    Let Δ be the energy change from recoloring v by q, and draw U uniformly from $[0, 1]$
    if $\Delta \leq 0$ or $U < e^{-\Delta/T}$ **then**
      Set $c(v) \leftarrow q$
    end if
    if $E(c) = 0$ **then**
      return c
    end if
    Set $T \leftarrow \alpha T$
  end for
end for
return FAILURE

Simulated annealing has previously been applied to graph coloring [16] and classical Ramsey problems [17]. We adapt it to polynomial distance graphs, where the objective is to find a four coloring with $E(c) = 0$. The difficulty is that a coloring can have positive energy even when every single vertex recoloring has $\Delta \geq 0$. The Metropolis rule [15] allows the search to move away from such local minima, while decreasing the temperature gradually favors colorings with fewer forbidden monochromatic pairs. We use independent random restarts and search for valid colorings on successively longer intervals. Kissat [10] and other deterministic search methods struggled on some of these instances, motivating our use of this probabilistic approach. Whenever the search finds a coloring with $E(c) = 0$, we verify it by checking every pair in $E_P(N)$. This establishes the rigorous lower bound $W(P(x); 4) \geq N + 1$.

5.2 Exact search

The same exact search is used for both Lemma 4.1 and Lemma 4.2.

The search uses the saturation degree rule of Brélaz [18]. At each step it chooses an uncolored vertex having the largest number of different colors among its colored neighbors. It then tries every possible color. Colors not yet used are equivalent under permutation, so only one new color needs to be tried. A branch is discarded as soon as an uncolored vertex has neighbors in all four colors.

Algorithm 2 Exact four color search

Input: Finite vertex set V and forbidden pair set E

```
function SEARCH(c)
  if every vertex is colored then
    return TRUE
  end if
  Choose an uncolored vertex v of maximum saturation degree
  Break ties by the number of uncolored neighbors
  Let T be the colors not used on colored neighbors of v
  Remove all but one color from T that has not yet been used
  for each q ∈ T do
    Give v color q
    if no uncolored vertex sees all four colors then
      if SEARCH(c) then
        return TRUE
      end if
    end if
    Uncolor v
  end for
  return FALSE
end function
return SEARCH(all vertices uncolored)
```

6 Conclusion

We determined nine exact four color polynomial van der Waerden numbers and established $W(ex^2 + fx; 4) = \Theta(|e| + |f|)$ for every nonzero polynomial $ex^2 + fx$. Our exact values are substantially smaller than the estimates given by general upper bounds, motivating the search for sharper bounds.

One immediate open problem is to determine the optimal constant C_4 such that

$$W\Big(ex^2 + fx; 4\Big) \leq C_4(|e| + |f|) + 1.$$

We established that $57 \leq C_4 \leq 64$.

More broadly, our work contributes to a research program of computing more exact polynomial van der Waerden numbers to better understand their growth. We aim to extend these methods to more colors and higher-degree polynomials, establish asymptotically tight bounds, and narrow the enormous gap between known lower and upper bounds. In particular, further exact values may help determine whether the growth of these numbers is closer to the exponential lower bounds or the much larger general upper bounds.

References

[1] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Archief voor Wiskunde* (2) **15** (1927), 212–216.

[2] S. Shelah, Primitive recursive bounds for van der Waerden numbers, *J. Amer. Math. Soc.* **1** (1988), 683–697.

[3] W. T. Gowers, A new proof of Szemerédi's theorem, *Geom. Funct. Anal.* **11** (2001), 465–588.

[4] W. Gasarch and B. Haeupler, Lower bounds on van der Waerden numbers: randomized and deterministic constructive, *Electron. J. Combin.* **18** (2011), P64.

[5] M. Kouril and J. L. Paul, The van der Waerden number $W(2,6)$ is 1132, *Experiment. Math.* **17** (2008), 53–61.

[6] M. Kouril, Computing the van der Waerden number $W(3,4) = 293$, *Integers* **12** (2012), Paper A46.

[7] D. Monroe, New lower bounds for van der Waerden numbers using distributed computing, *J. Combin. Math. Combin. Comput.* **128** (2026), 305–315. https://doi.org/10.61091/jcmcc128-19.

[8] M. Walters, Combinatorial proofs of the polynomial van der Waerden theorem and the polynomial Hales–Jewett theorem, *J. London Math. Soc.* (2) **61** (2000), 1–12.

[9] S. Shelah, A partition theorem, *Sci. Math. Jpn.* **56** (2002), 413–438.

[10] A. Biere, T. Faller, K. Fazekas, M. Fleury, N. Froleyks, and F. Pollitt, CaDiCaL, Gimsatul, IsaSAT and Kissat entering the SAT Competition 2024, *Proc. SAT Competition 2024*, 8–10.

[11] J. Barajas and O. Serra, Distance graphs with maximum chromatic number, *Discrete Mathematics and Theoretical Computer Science Proceedings*, AE (2005), 171–174. https://dmtcs.episciences.org/3391.

[12] V. Bergelson and A. Leibman, Polynomial extensions of van der Waerden's and Szemerédi's theorems, *J. Amer. Math. Soc.* **9** (1996), 725–753.

[13] W. Gasarch, C. Kruskal, J. Kruskal, and Z. Price, High school proofs for better bounds on the quadratic van der Waerden numbers, available at https://www.cs.umd.edu/~gasarch/GKKZP/gkkp.pdf.

[14] B. Canakci, H. Christenson, R. Fleischman, W. Gasarch, N. McNabb, and D. Smolyak, On SAT solvers and Ramsey-type numbers, arXiv:2312.01159, 2023.

[15] S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi, Optimization by simulated annealing, *Science* **220** (1983), 671–680.

[16] D. S. Johnson, C. R. Aragon, L. A. McGeoch, and C. Schevon, Optimization by simulated annealing: An experimental evaluation; Part II, graph coloring and number partitioning, *Operations Research* **39** (1991), 378–406.

[17] G. Exoo and M. Tatarevic, New lower bounds for 28 classical Ramsey numbers, *Electron. J. Combin.* **22** (2015), Paper 3.11. https://www.combinatorics.org/ojs/index.php/eljc/article/download/v22i3p11/pdf/.

[18] D. Brélaz, New methods to color the vertices of a graph, *Commun. ACM* **22** (1979), 251–256.